Deformed Poincaré Algebra and Field Theory
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Abstract

We examine deformed Poincaré algebras containing the exact Lorentz algebra. We impose constraints which are necessary for defining field theories on these algebras and we present simple field theoretical examples. Of particular interest is a case that exhibits improved renormalization properties.

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1 Introduction

Deformations of space-time and its symmetries have attracted a lot of attention recently [1]–[12]. The main reason from the particle physics perspective is that such deformed spaces or symmetries could be the basis to construct field theories with improved ultraviolet properties. This hope was motivated from the fact that q-deformations of space-time seem to lead to some lattice pattern which in turn could serve at least as some kind of regularization built in a theory that would be defined on such space-time.

Deformations of the Poincaré algebra (dPA) have been considered so far along three directions. The first consists of direct q-deformations of the Lorentz sub-algebra [1]–[3] along the lines prescribed by Drinfeld and Jimbo [11]. The second is based on the fact that the Poincaré algebra (PA) can be obtained by a Wigner-Inönü contraction of the simple anti-de Sitter algebra $O(3,2)$ [4, 5]. Then one first constructs the q-deformed $O_q(3,2)$ using the Drinfeld-Jimbo method and then does the contraction. In fact it was shown in ref.[8] that the same deformation can be obtained directly by considering general deformations of the commutation relations in the PA. Unfortunately the above deformations do not preserve the Lorentz algebra. Therefore it is natural to search for those dPAs that leave the Lorentz algebra unchanged in order to facilitate the quantization of the corresponding field theories. This motivation led us in ref.[10] to consider a third direction, namely deformations of the PA that leave the Lorentz algebra invariant.

In the present paper we continue our search for the appropriate dPAs that will allow us to construct field theories with impoved ultraviolet properties. We demand that the dPAs, in addition to leaving the Lorentz algebra invariant and giving the ordinary PA in low energies, should satisfy two more constraints. First, we require that there exists a tensor product of representations (coproduct) which is necessary in order to be able to go from the irreducible representations in the Hilbert space of quantum mechanics to the reducible representations in the Fock space of free quantum fields. A second requirement is that the representations of the dPA should be different from those of the ordinary PA, a usual property in q-groups [13], as a way to guarantee that the dPAs are well distinguished from ordinary PA and, in principle, with different physical implications. Finally we demonstrate using a scalar field theory defined on a specific dPA that indeed field theories with improved ultraviolet properties can be constructed.

2 The deformed Poincaré algebra

In ref.[10] it has been proposed to search for deformations of the PA that do not affect the Lorentz subalgebra.

The Lorentz algebra is a six-dimensional Lie algebra generated by the generators J_i, K_i of rotations and boosts correspondingly satisfying the following commutation relations

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \\ [J_i, K_j] &= i\epsilon_{ijk} K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k. \end{aligned} \tag{1}$$

Recall that by defining

$$N_i = \frac{1}{2}(J_i + iK_i)$$

one finds that N_i's and $N_i^\dagger$'s satisfy an $SU(2) \otimes SU(2)$ algebra. The enlargement of the Lorentz algebra to the Poincaré by including the energy-momentum generators (P_0, P_i) was proposed as a way to describe the quantum states of relativistic particles as unitary representations of the Poincareé group without using the wave equations [14]. One of the main points of ref.[10]was to show that this enlargement of the Lorentz algebra is not unique. Indeed, it was proposed to introduce a generalized set of commutation relations (as compared to the ordinary PA) for the generators (P_0, P_i, K_i) as follows

$$\begin{aligned} [K_i, P_0] &= i\alpha_i(P_0, \vec{P}), \\ [K_i, P_j] &= i\beta_{ij}(P_0, \vec{P}), \end{aligned} \tag{2}$$

where α_i, β_{ij} are functions of P_0, P_i. Then applying the Jacobi identities on the sets (J_i, K_i, P_0) and (J_i, K_j, P_k) it was found that the general form of α_i and β_{ij} is

$$\begin{aligned} \alpha_i(P_0, \vec{P}) &= \alpha(P_0, \vec{P}) P_i, \\ \beta_{ij}(P_0, \vec{P}) &= \beta(P_0, \vec{P})\delta_{ij} + \gamma(P_0, \vec{P}) P_i P_j. \end{aligned} \tag{3}$$

Assuming furthermore that there exists a Casimir invariant of the enlarged algebra of the form

$$f(P_0) - \vec{P}^2, \tag{4}$$

it was found that

$$\begin{aligned}\alpha_i(P_0,\vec{P}) &= \alpha(P_0)P_i,\\ \beta_{ij}(P_0,\vec{P}) &= \beta(P_0)\delta_{ij}.\end{aligned} \tag{5}$$

Moreover the closure of the enlarged algebra required that

$$\alpha(P_0)\beta'(P_0) = 1. \tag{6}$$

In this way a minimally deformed Poincaré algebra was constructed with commutation relations

$$\begin{aligned}[][J_i, J_j] &= i\epsilon_{ijk}J_k,\\ [J_i, K_j] &= i\epsilon_{ijk}K_k,\\ [K_i, K_j] &= -i\epsilon_{ijk}J_k,\\ [J_i, P_0] &= 0,\\ [J_i, P_j] &= i\epsilon_{ijk}P_k,\\ [P_0, P_i] &= 0,\\ [K_i, P_0] &= i\alpha(P_0)P_i,\\ [K_i, P_j] &= i\beta(P_0)\delta_{ij},\end{aligned} \tag{7}$$

where $\alpha(P_0), \beta(P_0)$ satisfy eq.(6). Note that the ordinary PA is obtained when $\alpha(P_0) = 1$ and $\beta(P_0) = P_0$.

The above dPA has two Casimir invariants. One corresponds to the length of the Pauli-Lubanski four-vector

$$W^2 = W_0^2 - \vec{W}\cdot\vec{W}, \tag{8}$$

where

$$\begin{aligned}W_0 &= \vec{J}\cdot\vec{P},\\ W_i &= \beta(P_0)J_i + \epsilon_{ijk}P_jK_k,\end{aligned} \tag{9}$$

with eigenvalues

$$W^2 = -\mu^2 s(s+1),$$

where $s = 0, 1/2, \ldots$ is the spin.

The other Casimir invariant of the dPA corresponds to the $(mass)^2$ of the ordinary PA and it is given by

$$\beta^2(P_0) - \vec{P} \cdot \vec{P} = \mu^2. \tag{10}$$

Let us also recall that the transformation $P_0 \to \beta(P_0)$ reduces the dPA to the ordinary PA for the set of generators $(J_i, K_i, P_i, \beta(P_0))$.

3 Constraints on the dPA parameter functions

From the construction of the dPA discussed above, it is clear that the functions $\beta(P_0)$ and, consequently, $\alpha(P_0)$ are not specified. An obvious physical requirement that these functions should satisfy is to let us obtain the ordinary PA as a limit of the dPA in low energies. Therefore we require that the low energy behaviour of $\beta(P_0)$ should be

$$\beta(P_0) \sim P_0$$

and therefore

$$\alpha(P_0) \sim 1.$$

One of our main aims is to define field theories on the constructed dPA hopefully with improved ultraviolet properties. A dPA with the Lorentz invariant subalgebra paves the way for an easy first quantization of such theories. Another requirement for construction of field theories is to be able to define the multiparticle states. In field theories defined on the ordinary PA the multiparticle states are constructed from the tensor product of one particle states. Here therefore we are looking for the corresponding "tensor" product which is usually called coproduct. In ref.[10] the $\beta(P_0)$ function was chosen to be

$$\beta(P_0) = M \sin(\frac{P_0}{M})$$

and the P_0 modulo periodicity was restricted to be in the interval $(-\frac{\pi M}{2}, \frac{\pi M}{2})$. This was a first attempt to improve the ultraviolet behaviour of theories defined on the dPA introducing an upper cut-off in the energy spectrum by

choosing a bounded function $\beta(P_0)$. However, when considering the additivity properties of energy $P_0^{(12)}$ of a system $S^{(12)}$ composed of two non-interacting systems $S^{(1)}, S^{(2)}$ some problems were found. Specifically, although the energy is conserved the energy $P^{(12)}$ was no longer the sum of the energies $P_0^{(1)}, P_0^{(2)}$ of the two subsystems $S^{(1)}, S^{(2)}$, respectively. So it was conjectured that the law of addition of the energies should be

$$\sin(\frac{P_0^{(1)}}{M}) + \sin(\frac{P_0^{(2)}}{M}) = 2\sin(\frac{P_0^{(12)}}{2M}).$$

The above conjecture, however, does not correspond to a true coproduct of the generator P_0 [1]. On the other hand, the above choice of $\beta(P_0)$ has also a positive aspect in the sense that the transformation $P_0 \rightarrow \beta(P_0)$ is not invertible since $\beta(P_0)$ is a multivalued function of P_0.

Here we would like to discuss two more alternatives as examples of the variety of existing possibilities. Let us first assume that the function $\beta(P_0)$ is of the form

$$\beta(P_0) = M\tanh^{-1}(\frac{P_0}{M}). \tag{11}$$

In this case the coproduct of the generators of the dPA are found to be

$$\begin{aligned}
\Delta(J_i) &= J_i \otimes 1 + 1 \otimes J_i, \\
\Delta(K_i) &= K_i \otimes 1 + 1 \otimes K_i, \\
\Delta(P_i) &= P_i \otimes 1 + 1 \otimes P_i, \\
\Delta(P_0) &= (P_0 \otimes 1 + 1 \otimes P_0)(1 \otimes 1 + \frac{P_0}{M} \otimes \frac{P_0}{M})^{-1}.
\end{aligned} \tag{12}$$

The coproduct (12) is determined by using the property that the transformation $P_0 \rightarrow \beta(P_0)$ transforms the dPA to the ordinary PA, which in the present case is invertible. Then the knowledge of the coproduct $\Delta(P_0)$ within PA easily gives us the $\Delta(P_0)$ for the dPA. At this point it should be emphasized that with the present choice of $\beta(P_0)$ the dPA is not a trivial redefinition of the ordinary PA. The reason is that the function $\alpha(P_0)$ is given by

$$\alpha(P_0) = \frac{1}{\beta'(P_0)} = 1 - \frac{P_0^2}{M^2}. \tag{13}$$

[1]We would like to thank L. Alvarez-Gaumé and O. Ogievetsky for pointing this to us

It is then clear that states with energy $P_0^2 \geq M^2$ are not representations of the dPA since the action of the boosts K_i on them would either leave unchanged or would reduce their energy. Therefore there is no one-to-one correspondence among the representations of dPA and PA as would be in the case of a trivial redefinition.

As a second example let us assume that the function $\beta(P_0)$ is

$$\beta(P_0) = M \tan^{-1}(\frac{P_0}{M}). \tag{14}$$

In this case again the coproduct is determined as before using the property that the transformation $P_0 \to \beta(P_0)$ takes the dPA to ordinary PA which is again invertible. The $\Delta(P_0)$ now becomes

$$\Delta(P_0) = (P_0 \otimes 1 + 1 \otimes P_0)(1 \otimes 1 - \frac{P_0}{M} \otimes \frac{P_0}{M})^{-1}. \tag{15}$$

Note however that the function $\alpha(P_0)$ now is

$$\alpha(P_0) = 1 + \frac{P_0^2}{M^2} \tag{16}$$

and thus it does not put any restrictions on the P_0's. Therefore there exists a one-to-one correspondence among the dPA and PA.

4 Scalar field theory on dPA

Here we shall examine a simple field theory such as $\lambda\phi^4$ on the dPA. The two examples for construction of multiparticle states discussed above will be examined separately. Let us consider the ordinary ϕ^4 in ordinary PA [15]. The Lagrangian is

$$L = L_0 + L_I$$

with

$$L_0 = \frac{1}{2}(\partial\phi_0)^2 - \frac{\mu_0^2}{2}\phi_0^2$$

and

$$L_I = -\frac{\lambda_0}{4!}\phi_0^4.$$

Recall that the self-energy graph at 1-loop is given by

$$-\Sigma(p^2) = -\frac{i\lambda_0}{2}\int d^4\ell \frac{i}{\ell_0^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \tag{17}$$

and it is quadratically divergent. Also the vertex corrections at 1-loop are given by

$$\Gamma(s) = (-\frac{i\lambda_0}{2})^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{(\ell_0 - p_0)^2 - (\vec{\ell} - \vec{p})^2 - \mu_0^2 + i\epsilon} \frac{i}{\ell_0^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \tag{18}$$

and $\Gamma(t), \Gamma(u)$ have similar expressions where

$$s = p^2 = (p_1 + p_2)^2\,,\; t = (p_1 - p_3)^2\,,\; u = (p_1 - p_4)^2$$

are the Mandelstam variables. The vertex corrections diverge logarithmically. The divergences of the self-energy and vertex corrections are removed by the well-known procedure of introducing counterterms and performing the renormalization program. We would like to examine whether the same theory when defined on the dPA has a better ultraviolet behaviour.

Given the form of $\beta(p_0)$ the L_0 part of the Lagrangian can be expressed locally in momentum space as

$$L_0 = \frac{1}{2}\left(\tilde{\phi}_0(p)(\beta^2(p_0) - \vec{p}^2)\tilde{\phi}_0(p) - \mu_0^2\tilde{\phi}_0(p)\tilde{\phi}_0(p)\right), \tag{19}$$

where $\tilde{\phi}_0(p)$ is the Fourier transform of $\phi_0(x)$. Therefore the propagator in the dPA is

$$\frac{i}{\beta^2(p_0) - \vec{p}^2 - \mu_0^2 + i\epsilon},$$

which with an appropriate choice of $\beta(p_0)$ can have more convergent behaviour for large p_0 as compared to the usual one

$$\frac{i}{p_0^2 - \vec{p}^2 - \mu_0^2 + i\epsilon}.$$

The calculation of 1-loop graphs reduces in determining integrals of the form

$$I(p^2) = \int d^4\ell \beta'(\ell_0) f(\beta^2(\ell_0) - \vec{\ell}^2, \beta^2(p_0) - \vec{p}^2) \tag{20}$$

i.e., integrals with dPA-invariant measure. Let us start with the case that $\beta(P_0)$ has the form (11). In this case the integrals resulting from 1-loop corrections will be

$$\begin{aligned} I_1 &= \int_{-M}^{M} d\ell_0 \beta'(\ell_0) \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2(\ell_0) - \vec{\ell}^2) \\ &= \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2 - \vec{\ell}^2) \end{aligned} \tag{21}$$

i.e., they are exactly the same as in the ordinary PA case. Therefore the present form of $\beta(P_0)$ and the corresponding dPA invariant measure does not improve the ultraviolet properties of the 1-loop corrections to the theory.

Let us then turn to our second example which exhibits a different behaviour. In this case $\beta(P_0)$ is given by eq(14). Then the integrals involved in the calculations of 1-loop corrections are

$$I_2 = \int_{-\infty}^{\infty} d\ell_0 \beta'(\ell_0) \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2(\ell_0) - \vec{\ell}^2) \tag{22}$$

$$= \int_{-M}^{M} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} f(\beta^2 - \vec{\ell}^2), \tag{23}$$

which are clearly more convergent (in view of the cut-off M) than the corresponding ones in the ordinary PA case. Therefore this choice of $\beta(P_0)$ provides us with an example of how one can improve the ultraviolet behaviour of a theory. The negative aspect of this particular choice of $\beta(P_0)$ is that the corresponding dPA has representations in one-to-one correspondence with ordinary PA and one would obtain the same results just by changing P_0 to $\beta(P_0)$ in ordinary PA. Therefore this last example cannot be considered seriously as having physical consequences but rather should be viewed as a regulator of the theory.

5 A satisfactory model

Here we present a choice of the function $\beta(P_0)$ which seems very promising since it satisfies the two new constraints we have demanded so far. Namely there exists a coproduct and the representations of the corresponding dPA are different from those of the ordinary PA. So we can construct a multiparticle state and the dPA under consideration is a distinct entity separate from

the ordinary PA. Then in principle the theory defined on this dPA could have different physical implications as compared to the same theory defined on ordinary PA. As we shall see, the one-loop self-energy and vertex corrections of a scalar field theory defined on this dPA has improved ultraviolet properties.

The chosen function is

$$\beta(P_0) = M \sin^{-1}(\frac{P_0}{M}). \tag{24}$$

The coproduct of the generators of the dPA are found to be

$$\begin{aligned}
\Delta(J_i) &= J_i \otimes 1 + 1 \otimes J_i \\
\Delta(K_i) &= K_i \otimes 1 + 1 \otimes K_i \\
\Delta(P_i) &= P_i \otimes 1 + 1 \otimes P_i \\
\Delta(P_0) &= P_0 \otimes \sqrt{1 - \frac{P_0^2}{M^2}} + \sqrt{1 - \frac{P_0^2}{M^2}} \otimes P_0.
\end{aligned} \tag{25}$$

As far as the representations are concerned, recall that the closure of the algebra requires that eq.(6) should hold which in turn implies that

$$\alpha(P_0) = \sqrt{1 - \frac{P_0^2}{M^2}}. \tag{26}$$

It is then clear that the representations with $P_0^2 \geq M^2$ are necessarily non-unitary while the energy spectrum for the unitary representations of the dPA lies in the interval $(-M, M)$. Therefore, since there is no unitary representations describing physical states of the dPA with $P_0^2 \geq M^2$, there is no one-to-one correspondence with the PA.

Turning to the one-loop self-energy and vertex corrections of the scalar theory we find that the self-energy graph becomes now

$$\begin{aligned}
-\Sigma(p^2) &= -\frac{i\lambda_0}{2} \int_{-\infty}^{\infty} d^4\ell \beta'(\ell_0) \frac{i}{\beta^2(\ell_0) - \vec{\ell}^2 - \mu_0^2 + i\epsilon} \\
&= -\frac{i\lambda_0}{2} \int_{-M\pi/2}^{M\pi/2} d\beta \int_{-\infty}^{\infty} d^3\vec{\ell} \frac{i}{\beta^2 - \vec{\ell}^2 - \mu_0^2 + i\epsilon}
\end{aligned} \tag{27}$$

which is linearly divergent instead of quadratically in the usual scalar theory defined on the ordinary PA. Correspondingly, the one-loop vertex corrections

take the form

$$
\begin{aligned}
\Gamma(s) &= (-\frac{i\lambda_0}{2})^2 \int \frac{d^4\ell}{(2\pi)^4} \beta'(\ell_0) \frac{i}{(\beta(\ell_0)-\beta(p_0))^2-(\vec{\ell}-\vec{p})^2-\mu_0^2+i\epsilon} \\
&\qquad \frac{i}{\beta^2(\ell_0)-\vec{\ell}^2-\mu_0^2+i\epsilon} \\
&= (-\frac{i\lambda_0}{2})^2 \int_{-M\pi/2}^{M\pi/2} \frac{d\beta d^3\vec{\ell}}{(2\pi)^4} \frac{i}{(\beta(\ell_0)-\beta(p_0))^2-(\vec{\ell}-\vec{p})^2-\mu_0^2+i\epsilon} \\
&\qquad \frac{i}{\beta^2(\ell_0)-\vec{\ell}^2-\mu_0^2+i\epsilon}.
\end{aligned} \tag{28}
$$

and similar forms take the $\Gamma(t), \Gamma(u)$ which are all convergent.

6 Discussion

The aim of the present paper was to present deformations of the Poincaré algebra that preserve the Lorentz sub-algebra requiring additional constraints that would pave the way for constructing realistic theories with improved ultraviolet properties. The new constraints that we have imposed are (i) the requirement of the existence of a coproduct of the representations of the dPA and (ii) the demand that there is no one-to-one correspondence among the representations of the dPA and ordinary PA which means that the two algebras are just homomorphic.

Concerning the first requirement, one may state, as a general rule, that a coproduct for the dPA always exists if $\beta(P_0)$ is an unbounded function of P_0. In that case, one may employ the homomorphism between the dPA and PA to pull back the coproduct of the PA into the dPA. Furthermore, if $\beta(P_0)$ is an odd function of P_0, the same homomorphism can also pull back the antipode of the PA into the dPA turning the latter into a cocommutative Hopf algebra. From the physical point of view, the cocommutativity of the dPA guarantees that the addition of observables for two systems $S^{(1)}$ and $S^{(2)}$ is independent of the order of addition. Recall for comparison that in the case of non-cocommutative algebras (quantum groups), the addition depends on the order (i.e., on the "labeling"). Turning to the second requirement it guarantees that the dPA is not a simple redefinition of ordinary PA.

Although the additional constraints are necessary in order to construct a field theory on a dPA which is not a trivial redefinition of the ordinary PA they do not guarantee that the theory has better ultraviolet behaviour. Therefore from this point of view they are necessary but not sufficient. On the other hand all the constraints considered so far, including the requirement for improved ultraviolet behaviour, cannot restrict in an appreciable manner the choices of the functions $\beta(P_0)$ which differentiate the various dPAs from each other.

We should emphasize that when a dPA satisfying all the above constraints is found it has very important physical consequences. First of all, such a dPA will be characterized by a non-trivial function $\beta(P_0)$ which certainly will result in observable deviations of the special theory of relativity. There exist already some analyses [16, 17] which put limits on the characteristic mass scale M appearing in general in $\beta(P_0)$ on dimensional grounds. For instance, according to ref.[17] the lowest bound consistent with experimental observations is $M_{min} \simeq 10^{12}$GeV.

Finally, a field theory with less divergences than the usual ones requires also less counterterms to cure them. This in turn means that the theory will have less free parameters to be fixed by experiment, or equivalently the theory will have more predictive power. It is expected then that the phenomenological constraints will provide us with enough information to restrict the possible choices of the functions $\beta(P_0)$. Moreover, it is fair to hope that genuine predictions on unkown parameters would emerge as a result of the above construction.

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